General Relativity For Tellytubbys

The Stress of Life That causes One To Get Tense

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Back to the Contents section

Overview

This section gives an outline of the *Stress-Energy* or *Energy-Momentum* Tensor. This little beastie is the thingymigigary that contains all the mass-energy and momentum of the 3 universes, and more to boot.

Stress-Energy/Energy-Momentum Tensor

The Stress Energy or Energy Momentum Tensor is an object containing information about all the mass, energy, and momentum of a system. Its covariant derivative results in the mass-energy and momentum conservation equations, for example the mass flow continuity equation and the Navier-Stokes equation of fluid mechanics all pop out in the wash.

From the SR section, we have

The 4-position

$$\mathbf{X} = [ct, x, y, z] = x^{\alpha}\mathbf{e}_{\alpha}$$

The 4-velocity

$$\mathbf{u} = [c\gamma, \gamma\dot{x}, \gamma\dot{y}, \gamma\dot{z}]$$

$$u^0 \equiv c\frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \gamma c$$

$$u^{\alpha} = \frac{dx^{\alpha}}{d\tau} = \gamma\frac{dx}{dt} = \gamma v^{\alpha}$$, alpha =1,2,3

The 4-momentum

$$\mathbf{p} = m\mathbf{u}$$

$$\mathbf{p} = [\frac{E}{c}, \gamma m v^x, \gamma m v^y, \gamma m v^z]$$

First, a refresh. This assumes some prior fluid mechanics.

Given some dust collection or fluid substance, with zero pressure, crossing some surface da it should be seen that:

an element of mass flow in unit time is dm=ρv.da

The total mass flow rate out of a volume contained by that surface is therefore

$$\dot{m} = \iint \rho \mathbf{v}.\mathbf{da}$$

The total mass in a volume is given by

$$m = \iiint \rho dv$$

Therefore, what flows out from the volume must equal what crosses the volume's enclosing surface.

$$\iiint \rho dv = -\oiint_a \rho \mathbf{v}.\mathbf{da}$$ And using Mr. Gauss or Mr. Green…

$$\frac{\partial}{\partial t}\iiint \rho dv = -\iiint \nabla.(\rho \mathbf{v})dv$$

Hence:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0$$

This leads to a definition of the energy-momentum tensor as the flux of 4-momemtum across a surface, but this is far to complicated for us Teletubbys so lets wave a bit to Po.

Lets imagine a parallelepiped (slant sided box) and its faces, with stuff flowing through the faces. The force (vector) acting at any face will be a function of the area, direction to that area and on object, that characterizes how all stuff is flowing about. This object is the stress tensor i.e.

$$\Delta \mathbf{F} = \boldsymbol{\sigma}.\mathbf{n}\Delta a$$

So, one has an object that is a product with the normal (vector) of the surface which must give a vector as a result, therefor that object must be a tensor of rank 2

This can also be expressed as

$$\mathbf{F} = \iint \boldsymbol{\sigma}.\mathbf{n}da$$

In component form we can write

$$\Delta F^i = \sigma^{ij}\Delta a_j$$

or as a definition of the stress tensor

$$\sigma^{ij} = \frac{\Delta F^i}{\Delta a_j}\Big|_{\Delta a_j \to 0}$$

For our generic piece of stuff floating around, let calculate in terms of momentum, cos we know what the momentum density flow, per unit time, is from above, i.e. volume times density is mass.

$$\sigma^{ij} = \frac{\Delta v}{\Delta a^j}.\frac{(\rho v^i)}{\Delta t}$$

$$\sigma^{ij} = \frac{\Delta x^i \Delta x^j \Delta x^k}{\Delta a^j} . \frac{(\rho v^i)}{\Delta t}$$

$$\sigma^{ij} = \frac{\Delta x^i \Delta x^j \Delta x^k}{\Delta x^i \Delta x^k} . \frac{(\rho v^i)}{\Delta t}$$

$$\sigma^{ij} = \frac{\Delta x^j}{\Delta t} .(\rho v^i)$$

$$\sigma^{ij} = v^j .(\rho v^i)$$

or

$$\sigma^{ij} = \rho v^i v^j$$

which is just the tensor product of velocities. i.e.

$$\boldsymbol{\sigma} = \rho \mathbf{v} \otimes \mathbf{v}$$

And now generalizing this tensor to a 4-D space-time tensor we get

$$\mathbf{T} = \rho \mathbf{u} \otimes \mathbf{u}$$

As the stress-energy or energy-momentum, depending on which book you read, tensor

So, lets work out

$$T^{0j}{}_{,j} = 0$$

$$(\rho u^0 u^j)_{,j} = 0$$

for low velocities, i.e. gamma =1,

$$(\rho u^0 u^j)_{,j} = (\rho c u^j),_j = 0 = (\rho u^j),_j$$

$$(\rho c)_{,0} + (\rho v^1)_{,1} + (\rho v^2)_{,2} + (\rho v^2)_{,3} = 0$$

$$\frac{\partial (c\rho)}{\partial x^0} + \frac{\partial (\rho v^1)}{\partial x^1} + \frac{\partial (\rho v^2)}{\partial x^2} + \frac{\partial (\rho v^3)}{\partial x^3} = 0$$

$$\frac{\partial (c\rho)}{\partial ct} + \frac{\partial (\rho v^x)}{\partial x} + \frac{\partial (\rho v^y)}{\partial y} + \frac{\partial (\rho v^z)}{\partial z} = 0$$

$$\frac{\partial (\rho)}{\partial t} + \nabla .(\rho \mathbf{v}) = 0$$

So, one recovers the conservation of mass equation.

It can also be seen from inspection that the 0th row of the tensor contains the total energy and momentum density of the stuff, i.e.

$$T^{00} = \rho(\gamma c)(\gamma c) = c^2(\rho\gamma)\gamma = E_{dent}$$

Noting the contraction of the volume element as well as the relativistic momentum density

$$T^{0i} = c(\rho\gamma v^i)\gamma = cp^i \text{ where } \mathrm{p}^{\mathrm{i}} \text{ is the momentum density}$$

$$T^{ij} = \sigma^{ij} = relativistic\ \frac{f^i}{a^j} \text{ which also is the flux of momentum in unit time accross a surface}$$

In general, with this definition it can be shown that

$$\nabla.\mathbf{T} = T^{ij}{}_{;j} = 0$$

Perfect Fluid

With a bit of piddling about the energy momentum tensor for a perfect fluid can be derived as:

$$\mathbf{T} = p\mathbf{g} + (\rho + \frac{p}{c^2})\mathbf{u} \otimes \mathbf{u}$$

Which, I will leave till later.

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http://www.kevinaylward.co.uk/gr/index.html

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