

# On the Validity of Orthogonally Decomposed Perturbations in Phase Noise Analysis

Alper Demir      Jaijeet Roychowdhury  
{alpdemir, jaijeet}@research.bell-labs.com

Bell Laboratories  
Murray Hill, New Jersey, USA

## Abstract

When an autonomous system, i.e., an oscillator, in a stable orbit in the state space, is disturbed by a perturbation that is *along* the tangent to the limit cycle, it stays in its unperturbed orbit though with a phase error with respect to the unperturbed oscillator. It might seem intuitive to think that, when the perturbation is *orthogonal* to the tangent to the limit cycle, the oscillator will deviate from its orbit but will eventually return to it with a zero phase error when the perturbation is removed. We show that this seemingly intuitive observation is not correct. We, in fact, show that if the perturbation lies in a certain transversal hyperplane, which is in general *obliquely slanted* to the tangent, the phase error in steady-state will be zero when the perturbation is removed. We provide an exact characterisation of this obliquely slanted transversal hyperplane, and present a simple example to illustrate the theory. We also show that one can decompose perturbations in general into two oblique components, one of which causes only phase errors, and the other only orbital deviations. Once the perturbation is removed, the phase error in general persists, but the orbital deviations decay to zero. Based on the oblique decomposition, we derive a nonlinear differential equation for phase error. Considering random perturbations, we provide a stochastic characterisation of the phase error and the oscillator output.

We then consider the question: Even if the orthogonal decomposition of the perturbation is not strictly valid, (i.e., the component that is orthogonal to the tangent, as well as the one that is parallel to it, causes phase errors), is it approximately correct? Can it provide approximately correct characterisations of phase noise in electrical oscillators with small perturbations (i.e., noise)? We show that the answer to the above questions is negative, which we illustrate with some examples.

Furthermore, we discuss the implications of assuming that the relationship between the phase error and the perturbation is linear. We show that this assumption leads to nonphysical results (even when the perturbation is small), and it is unjustified and incorrect to use superposition to calculate the phase error due to two perturbations applied at the same time by calculating the phase errors arising from them separately and then summing them up.

## 1 Introduction

In a recent publication [DMR97], we presented a unifying theory and numerical methods for practical characterisation of phase noise in oscillators. We obtained an exact *nonlinear* differential equation for the phase error resulting from perturbations to an oscillator in a stable limit cycle, and were able to characterise the temporal and spectral properties of the phase error under random perturbations. A key feature of our phase noise analysis was the decomposition of the applied perturbation into two parts:

- one along the tangent to the limit cycle of the unperturbed oscillator, which causes a persisting phase error, and
- another in a transversal hyperplane which is in general *obliquely slanted* to the tangent. This transversal component does *not* result in a persisting phase error.

At first sight, the necessity of decomposing the perturbation obliquely so that the transversal component does not cause persisting phase errors may appear puzzling, especially when the dynamics of the oscillator's limit cycle are interpreted mechanically, e.g., analogous to a stone swung around a string. An alternative approach, decomposing the perturbation *orthogonally*, appears more intuitive to many, the notion being that perturbations orthogonal to the tangent will not impact the "speed", hence phase error, of the system. This course has been proposed by the authors in [HL97] and forms the backbone of their analysis of the phase noise problem. They derive a *linear* equation for phase error based on the orthogonal decomposition and an unjustified superposition argument which assumes that the relationship between the phase error and the perturbation is linear. Kärtner in [Kär90] also derives a *linear* equation for phase error, even though his derivation is based on the correct oblique decomposition. At first sight, the linearisation approximation might seem to be justifiable considering that the perturbations are "small".

However, when one considers random perturbations, i.e., noise (a small perturbation) one arrives at *nonphysical* results using linearisation.

In Section 2, we start with the discussion of some preliminary notions relating to the dynamics of autonomous oscillators. In Section 3, we compare orthogonal decomposition with oblique decomposition explicitly, from both theoretical and practical perspectives. We consider the questions:

1. Is the orthogonal decomposition valid in general?
2. Even if it is not strictly valid, can it provide approximately correct results and intuition for practical oscillator designs?

We show that the answer to both questions is negative. In Section 4, for the case of random perturbations, i.e., noise, we discuss the nonphysical results one obtains assuming that the phase error is linearly related to the perturbation.

In Section 5, we analyse several examples to illustrate the invalidity of the orthogonal decomposition and correctness of the oblique one. We consider a simple example with momentary impulsive perturbations which illustrates how a perturbation that is orthogonal to the tangent to the limit cycle *can* create phase errors; and a perturbation that is *not* orthogonal to the tangent to the limit cycle, but that does *not* have a component along the tangent when decomposed using an *oblique* basis (the tangent vector is a component of this basis), does not result in a phase error. For electrical oscillators, we demonstrate that while, in certain cases, phase noise characterisation results predicted by orthogonal decomposition can indeed be close to the correct values obtained by oblique decomposition, for other practical circuits, it can predict results off by as much as 50 dBc/Hz.

## 2 Orbital stability and Floquet theory

The dynamics of any autonomous system without undesired perturbations can be described by a system of differential equations:

$$\dot{x} = f(x) \tag{1}$$

where  $x \in \mathbb{R}^n$  and  $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . We assume that  $f(\cdot)$  satisfies the conditions of the Picard-Lindelöf existence and uniqueness theorem for initial value problems.

Let  $x_s(t)$  be a non-constant  $T$ -periodic solution of the autonomous system (1), which forms a closed orbit in the  $n$ -dimensional space. Note that  $x_s(t - t_0)$  is also a non-constant  $T$ -periodic solution of (1) for arbitrary  $t_0 \in \mathbb{R}$ . The initial values  $x_s(0)$  and  $x_s(-t_0)$  can be arbitrarily close (if  $|t_0|$  is small enough), and still  $x_s(t) - x_s(t - t_0)$  does not tend to zero as  $t$  tends to infinity.

**Definition 2.1** Let the path  $\gamma$  of the  $T$ -periodic solution  $x_s(t)$  be

$$\gamma := \{x \in \mathbb{R}^n : x = x_s(t), t \in \mathbb{R}_+\}$$

Note that  $x_s(t)$  and  $x_s(t - t_0)$  have the same path  $\gamma$ .

**Definition 2.2 (Orbital Stability)** The solution  $x_s(t)$  of (1) is said to be orbitally stable if for every  $\varepsilon > 0$  there exists a  $\delta(\varepsilon)$  such that if the distance of the initial value  $x(0) = x_0$  from the path  $\gamma$  of  $x_s(t)$  is less than  $\delta(\varepsilon)$ , i.e.,  $\text{dist}(x_0, \gamma) < \delta(\varepsilon)$ , then the solution  $\phi(t, x_0)$  of (1) that assumes the value  $x_0$  at  $t = 0$  satisfies

$$\text{dist}(\phi(t, x_0), \gamma) < \varepsilon$$

for  $t \geq 0$ .

If the solution  $x_s(t)$  is orbitally stable, then each solution with the same path  $\gamma$ , i.e., every solution  $x_s(t + \alpha)$  for  $\alpha \in \mathbb{R}$ , is orbitally stable too.

**Definition 2.3 (Asymptotic Orbital Stability)** The solution  $x_s(t)$  of (1) is said to be asymptotically orbitally stable if it is orbitally stable, and if a  $\delta > 0$  exists such that  $\text{dist}(x_0, \gamma) < \delta$  implies

$$\text{dist}(\phi(t, x_0), \gamma) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

**Definition 2.4 (Asymptotic Phase Property)** The solution  $x_s(t)$  is said to have the asymptotic phase property if a  $\delta > 0$  exists such that to each initial value  $x_0$  satisfying  $\text{dist}(x_0, \gamma) < \delta$  there corresponds an asymptotic phase  $\alpha(x_0) \in \mathbb{R}$  with the property

$$\lim_{t \rightarrow \infty} |\phi(t, x_0) - x_s(t + \alpha(x_0))| = 0$$

We consider systems that have a non-trivial periodic solution  $x_s(t)$  which is asymptotically orbitally stable, and has the asymptotic phase property.

Consider the  $n$ -dimensional homogeneous linear system of differential equations

$$\dot{w} = A(t)w(t) \quad (2)$$

where the Jacobian

$$A(t) = \left. \frac{\partial f(x)}{\partial x} \right|_{x_s(t)} \quad (3)$$

is  $T$ -periodic. The state transition matrix [DMR97] for (2) is given by

$$\Phi(t, s) = U(t) \exp(D(t - s))V(s) \quad (4)$$

where  $U(t)$  is a  $T$ -periodic nonsingular matrix,  $V(t) = U^{-1}(t)$  and  $D = \text{diag}[\mu_1, \dots, \mu_n]$ , where  $\mu_i$  are called the *Floquet (characteristic) exponents*.  $\exp(\mu_i T)$  are called the *characteristic multipliers* [DMR97].

**Remark 2.1 ([DMR97])** Let  $u_i(t)$  be the columns of  $U(t)$  and  $v_i^T(t)$  be the rows of  $V(t) = U^{-1}(t)$ . Then  $\{u_1(t), u_2(t), \dots, u_n(t)\}$  and  $\{v_1(t), v_2(t), \dots, v_n(t)\}$  both span  $\mathbb{R}^n$  and satisfy the biorthogonality conditions  $v_i^T(t)u_j(t) = \delta_{ij}$  for every  $t$ . Note that, in general,  $U(t)$  itself is not an orthogonal matrix.

**Remark 2.2 ([DMR97])**

- The unperturbed oscillator (1) has a non-trivial  $T$ -periodic solution  $x_s(t)$  if and only if the number 1 is a characteristic multiplier of (2), or equivalently, one of the Floquet exponents satisfies  $\exp(\mu_i T) = 1$ .
- The time-derivative of the periodic solution  $x_s(t)$  of (1), i.e.,  $\dot{x}_s(t)$ , is a solution of the homogeneous system in (2).

We now state an important result:

**Remark 2.3** One can show that if 1 is a characteristic multiplier, and the remaining  $n - 1$  Floquet exponents satisfy  $|\exp(\mu_i T)| < 1$ ,  $i = 2, \dots, n$ , then the periodic solution  $x_s(t)$  of (1) is asymptotically orbitally stable, and it has the asymptotic phase property. Moreover, if any of the Floquet exponents satisfy  $|\exp(\mu_i T)| > 1$ , then the solution  $x_s(t)$  is orbitally unstable.

**Remark 2.4** Without loss of generality, one can choose  $\mu_1 = 0$  and  $u_1(t) = \dot{x}_s(t)$ . With  $u_1(t) = \dot{x}_s(t)$ , we have  $v_1^T(t)\dot{x}_s(t) = 1$  and  $v_1^T(t)u_j(t) = 0$ ,  $j = 2, \dots, n$ .  $v_1(t)$  will play an important role in our treatment later.

### 3 Nonlinear perturbation analysis

We are interested in the response of asymptotically orbitally stable systems in (1) to a small state-dependent perturbation of the form  $B(x)b(t)$  where  $B(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p}$  and  $b(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^p$ . Hence the perturbed system is described by

$$\dot{x} = f(x) + B(x)b(t) \quad (5)$$

We rewrite (5) with the (small) perturbation  $B(x)b(t)$  split into two small parts  $b_1(x, t)$  and  $\tilde{b}(x, t)$ :

$$\dot{x} = f(x) + b_1(x, t) + \tilde{b}(x, t) \quad (6)$$

Next, we will choose the first perturbation term  $b_1(x, t)$  in such a way that its effect is to create only *phase errors* to the unperturbed solution  $x_s(t)$ . In other words, for a certain choice of  $b_1(x, t)$ , we will show that the equation

$$\dot{x} = f(x) + b_1(x, t) \quad (7)$$

is solved by

$$x_p(t) = x_s(t + \alpha(t)) \quad (8)$$

for a certain function  $\alpha(t)$ , called the *phase deviation*.

We split the perturbation  $B(x)b(t)$  into  $b_1(x, t)$  and  $\tilde{b}(x, t)$  and define the phase deviation  $\alpha$  as follows:

**Definition 3.1**

1. Let

$$b_1(x, t) = c_1(x, \alpha(t), t)u_1(t + \alpha(t)), \quad \text{and} \quad (9)$$

$$\tilde{b}(x, \alpha(t), t) = B(x)b(t) - b_1(x, t) \quad (10)$$

where  $c_1(x, \alpha(t), t)$  is a scalar.

2. Define  $\alpha(t)$  by

$$\frac{d\alpha(t)}{dt} = c_1(x_s(t), \alpha(t), t), \quad \alpha(0) = 0 \quad (11)$$

We will discuss the exact specification of  $c_1(x, \alpha, t)$  later. Note that  $b_1(x, t)$  is obtained by scaling the vector  $u_1(t + \alpha(t)) = \dot{x}_s(t + \alpha(t))$  with the scalar  $c_1(x, \alpha, t)$ .

**Lemma 3.1**  $x_p(t) = x_s(t + \alpha(t))$  solves (7).

**Proof:** Substituting  $x_s(t + \alpha(t))$  in (7) and using  $\dot{x}_s(t) = u_1(t)$  we obtain

$$\dot{x}_s(t + \alpha(t))(1 + \dot{\alpha}(t)) = f(x_s(t + \alpha(t))) + c_1(x_s(t + \alpha(t)), \alpha(t), t)u_1(t + \alpha(t))$$

$$\Rightarrow \dot{\alpha}(t)u_1(t + \alpha(t)) = c_1(x_s(t + \alpha(t)), \alpha(t), t)u_1(t + \alpha(t))$$

Note that  $\alpha(t)$  and  $c_1(t)$  are scalars while  $u_1(t) = \dot{x}_s(t)$  is a vector. Also for any  $t$ , all the entries of  $\dot{x}_s(t)$  and hence  $u_1(t)$  cannot be simultaneously zero otherwise the oscillator will cease to oscillate. Hence  $\dot{\alpha}(t) = c_1(x_s(t + \alpha(t)), \alpha(t), t)$ . ■

With Lemma 3.1, we have shown that the  $b_1(x, t)$  component of the perturbation causes deviations only *along* the limit cycle, i.e., phase deviations. This result is intuitively appealing, since we restricted  $b_1(x, t)$  (which has a time-varying direction) to be parallel to the time-shifted tangent vector  $\dot{x}_s(t + \alpha(t)) = u_1(t + \alpha(t))$  to the limit cycle, hence the perturbation  $b_1(x, t)$  “pushes” or “pulls” the system only along the limit cycle resulting in phase deviations only. We would like to emphasize the fact that Lemma 3.1 is valid for any choice of the scalar  $c_1(x, \alpha, t)$  in (9). We have not specified  $c_1(x, \alpha, t)$  yet, which we discuss next.

What does the rest of the perturbation, i.e.,  $\tilde{b}(x, t)$ , do? To answer this question, we have to, first, fully specify  $c_1(x, \alpha(t), t)$  in (9).

**Remark 3.1**

1. In Definition 3.1.1, where we decomposed the perturbation into two components, we specified only one of the basis vectors, i.e.,  $u_1(t + \alpha(t))$ , and chose the first component to be in the span of this vector. To fully specify the decomposition, one needs to specify the rest of the basis vectors, in the span of which the second perturbation component lies.
2. We would like to choose  $c_1(x, \alpha(t), t)$  (hence  $\alpha(t)$ ), or specify the rest of the basis vectors, in such a way that the effect of  $\tilde{b}(x, t)$  is to cause only “small” (non-growing) excursions (which we call orbital deviations) from the time shifted limit cycle  $x_s(t + \alpha(t))$ . If the perturbation does not persist, i.e., if  $b(t) = 0$  for  $t > t_c$  for some cut-off time  $t_c$ , we would like the orbital deviations caused by  $\tilde{b}(x, t)$  to decay to 0, i.e., we would like  $x_s(t + \alpha(t))$  to solve (5) as  $t \rightarrow \infty$ .

In [DMR97], we describe an exact specification of  $c_1(x, \alpha(t), t)$  (hence  $\alpha(t)$ ) in (9) and the rest of the basis vectors for the decomposition of the perturbation, and prove that Remark 3.1.2 is realized with this decomposition. For convenience, we reproduce our results here. Choose

$$c_1(x, \alpha(t), t) = v_1^T(t + \alpha(t))B(x)b(t) \quad (12)$$

and hence  $\alpha(t)$  is defined by

$$\frac{d\alpha(t)}{dt} = v_1^T(t + \alpha(t))B(x_s(t + \alpha(t)))b(t), \quad \alpha(0) = 0 \quad (13)$$

where  $v_1(t)$  was defined in Remark 2.1 and Remark 2.4. Then,  $\tilde{b}(x, \alpha(t), t)$  is given by

$$\tilde{b}(x, \alpha(t), t) = B(x)b(t) - b_1(x, t) = B(x)b(t) - v_1^T(t + \alpha(t))B(x)b(t)u_1(t + \alpha(t)) = \sum_{i=2}^n c_i(x, \alpha(t), t)u_i(t + \alpha(t)) \quad (14)$$

where the scalars

$$c_i(x, \alpha(t), t) = v_i^T(t + \alpha(t))B(x)b(t) \quad (15)$$

### Remark 3.2

- The perturbation  $B(x)b(t)$  is decomposed into components using the basis  $\{u_1(t + \alpha(t)), \dots, u_n(t + \alpha(t))\}$  of Remark 2.1 and Remark 2.4. The coefficients  $c_i(x, \alpha(t), t)$  of the components of  $B(x)b(t)$  along the basis vectors  $u_i(t + \alpha(t))$  were obtained above using the biorthogonality relationships between the basis vectors in  $\{u_1(t + \alpha(t)), \dots, u_n(t + \alpha(t))\}$  and in  $\{v_1(t + \alpha(t)), \dots, v_n(t + \alpha(t))\}$  from Remark 2.1.
- The first vector in the basis  $\{u_1(t + \alpha(t)), \dots, u_n(t + \alpha(t))\}$  is  $u_1(t + \alpha(t)) = \dot{x}_s(t + \alpha(t))$  that is tangent to the limit cycle.  $b_1(x, t)$  is the component of the perturbation along this first basis vector.
- The rest of the perturbation  $\tilde{b}(x, \alpha(t), t) = B(x)b(t) - b_1(x, t)$  lies in the span of the rest of the basis vectors, i.e., in the span of  $\{u_i(t + \alpha(t)), i = 2, \dots, n\}$ . The basis  $\{u_1(t + \alpha(t)), \dots, u_n(t + \alpha(t))\}$  is, in general, not an orthogonal basis. Moreover,  $u_1(t)$  is, in general, not orthogonal to  $\text{span}\{u_i(t + \alpha(t)), i = 2, \dots, n\}$ .

By solving the nonlinear equation in (13), one can calculate the phase error (time-varying in general) for a given perturbation.

**Remark 3.3** The right-hand-side (RHS) of the differential equation (13) for the phase error  $\alpha$  is nonlinear. Thus, one can not use superposition to calculate the phase error due to several perturbations, i.e., one can not calculate the phase errors due to two perturbations separately and then sum them up to obtain the phase error due to the two perturbations applied at the same time.

### 3.1 Non-persisting perturbations

We now consider perturbations that do not persist, i.e.,

$$b(t) = 0 \quad \text{for } t > t_c \quad (16)$$

for some cut-off time  $t_c$ . After the perturbations cuts off at  $t = t_c$ , the orbital deviations due to  $\tilde{b}(x, t)$  decay to 0, and the solution of (5) returns to the unperturbed limit cycle, though with a phase error. We can calculate this phase error using (13). From (13) and (16), we have

$$\frac{d\alpha(t)}{dt} = 0 \quad \text{for } t > t_c \quad (17)$$

and hence

$$\alpha(t) = \alpha(t_c) = \alpha_c \quad \text{for } t > t_c \quad (18)$$

$\alpha_c$  can be calculated by numerically solving the nonlinear differential equation (13) as an initial value problem from  $t = 0$  to  $t = t_c$ .

**Remark 3.4**  $x_s(t + \alpha_c)$  solves (5) (with (16)) as  $t \rightarrow \infty$ . If the perturbation  $B(x_s(t + \alpha(t)))b(t)$  is orthogonal to  $v_1(t + \alpha(t))$  for  $0 \leq t \leq t_c$ , then  $\alpha_c = 0$  (In fact,  $\alpha(t) = 0$  for  $0 \leq t \leq t_c$ ). In general,  $\alpha_c \neq 0$  even if the perturbation is orthogonal to the tangent  $u_1(t) = \dot{x}_s(t)$  to the limit cycle for  $0 \leq t \leq t_c$ .

Next, we analyze a special case of perturbations that do not persist, i.e.,  $\delta$ -function perturbations.

### 3.2 $\delta$ -function perturbations

Let us now consider the special case of a  $\delta$ -function perturbation in the form

$$\tilde{b}\delta(t) \quad (19)$$

where  $\tilde{b}$  is a vector. We decompose  $\tilde{b}\delta(t)$  as in Definition 3.1.1:

$$\tilde{b}\delta(t) = b_1\delta(t) + \tilde{b}\delta(t) = c_1u_1(0)\delta(t) + \tilde{b}\delta(t) \quad (20)$$

If we substitute  $c_1(x_s(t), \alpha(t), t) = c_1\delta(t)$  in (11) and solve it for  $\alpha(t)$ , we obtain

$$\alpha(t) = \begin{cases} 0 & \text{if } t = 0 \\ c_1 & \text{if } t > 0 \end{cases}$$

From Lemma 3.1, we conclude that  $x_s(t + c_1)$  solves

$$\dot{x} = f(x) + b_1\delta(t) = f(x) + c_1u_1(0)\delta(t) = f(x) + c_1\dot{x}_s(0)\delta(t)$$

for  $t > 0$ . The  $c_1u_1(0)\delta(t)$  component of the perturbation  $\tilde{b}\delta(t)$  causes a sudden phase deviation  $c_1$  along the limit cycle, without causing any orbital deviation. What does the rest of the perturbation, i.e.,  $\tilde{b}\delta(t)$ , do?

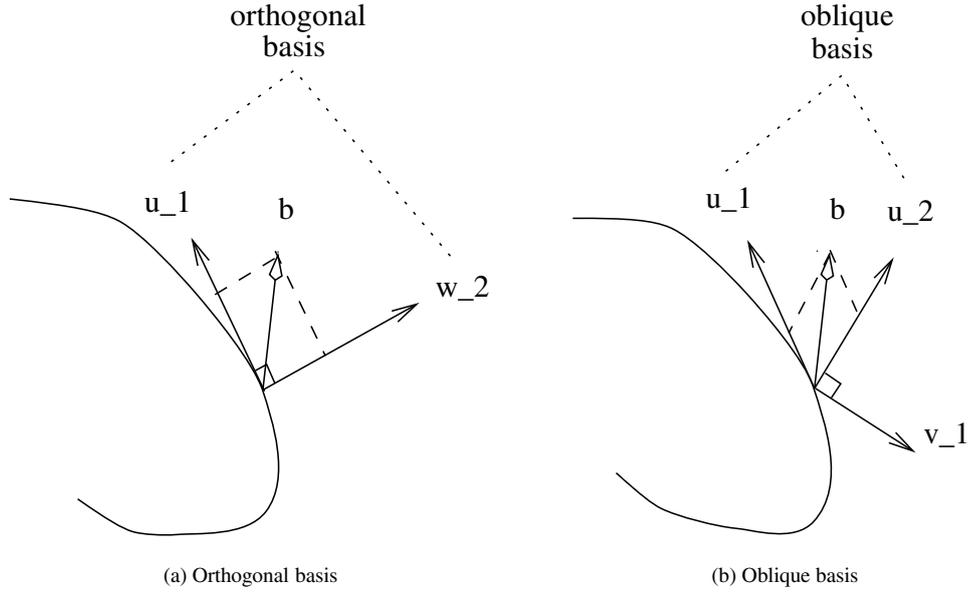


Figure 1: Decomposition of the perturbation

**Remark 3.5** If we choose  $c_1$  as in (12), i.e.,

$$c_1 = v_1^T(0) \bar{b} \quad (21)$$

then the effect of  $\bar{b} \delta(t)$  is to cause only decaying orbital deviations from the time shifted limit cycle  $x_s(t + c_1)$ , i.e.,  $x_s(t + c_1)$  solves

$$\dot{x} = f(x) + \bar{b} \delta(t) = f(x) + b_1 \delta(t) + \tilde{b} \delta(t)$$

as  $t \rightarrow \infty$ .

Thus, given the perturbation  $\bar{b} \delta(t)$ , we can calculate the asymptotic phase error using (21).

Authors in [HL97] conjecture that if  $c_1$  in (20) is chosen in such a way that  $\bar{b}$  is orthogonal to the tangent vector  $u_1(0) = \dot{x}_s(0)$ , then the  $\tilde{b} \delta(t)$  component of the perturbation will not cause any phase errors. This result is in contradiction with our result. In the appendix of [HL97], the authors provide a proof for this conjecture. Unfortunately, the argument in their proof is flawed, and the result they prove is invalid. In Section 5.2, we will illustrate the invalidity of their result (and the correctness of ours) with a simple example. Please see Figure 1 for an illustration of oblique and orthogonal decompositions.

## 4 Random perturbations

In [DMR97], we consider random perturbations, i.e., when the perturbation  $b(t)$  in (5) is a vector of uncorrelated white noise sources. We treat (13) as a nonlinear stochastic differential equation and arrive at some non-trivial results summarized as follows:

**Remark 4.1**

1.  $\alpha(t)$  described by the stochastic differential equation (13), becomes, asymptotically with  $t$ , a Gaussian random variable with mean  $\mu(t) = m$  and variance  $E[\alpha^2(t)] = \sigma^2(t) = ct$ , where

$$c = \frac{1}{T} \int_0^T v_1^T(\tau) B(x_s(\tau)) B^T(x_s(\tau)) v_1(\tau) d\tau \quad (22)$$

2. The time-shifted output of the oscillator, i.e.  $x_s(t + \alpha(t))$ , with the  $\alpha(t)$  characterised as above, becomes a stationary stochastic process asymptotically with time and its power spectral density is given by

$$S(f) = 2 \sum_{i=-\infty}^{\infty} X_i X_i^* \frac{f_0^2 i^2 c}{\pi^2 f_0^4 i^4 c^2 + (f + i f_0)^2} \quad (23)$$

where  $f_0$  is the oscillation frequency,  $c$  is given in (22), and the  $X_i$  are the Fourier series coefficients in the expansion of the periodic  $x_s(t)$ .

In Section 3, we discussed the invalidity (to be illustrated in Section 5.1) of orthogonal (i.e., one along the tangent to the limit cycle and the other in a transversal hyperplane that is orthogonal to the tangent) decomposition in calculating the phase error due to perturbations. Now, we would like to answer the following question: Even if orthogonal decomposition is not strictly valid, can it provide approximately correct results for the characterisation of phase noise in practical oscillators and provide intuition to designers? To answer this question, we now derive an equation for the phase error  $\alpha(t)$  using the invalid orthogonal decomposition as an alternative to the one we derived in (13) using the valid oblique decomposition. Instead of using  $\{u_1(t + \alpha(t)), \dots, u_n(t + \alpha(t))\}$  as the basis to decompose the perturbation  $B(x)b(t)$ , we use

$$\{u_1(t + \alpha(t)), w_2(t + \alpha(t)), \dots, w_n(t + \alpha(t))\}$$

where

$$u_1^T(t) w_i(t) = 0 \quad \text{for } i = 2, \dots, n \quad (24)$$

and

$$u_1(t) = \dot{x}_s(t)$$

We have

$$B(x)b(t) = c_1(x, \alpha(t), t)u_1(t + \alpha(t)) + \sum_{i=2}^n c_i(x, \alpha(t), t)w_i(t + \alpha(t))$$

where

$$c_1(x, \alpha(t), t) = \frac{u_1^T(t + \alpha(t)) B(x)b(t)}{u_1^T(t + \alpha(t)) u_1(t + \alpha(t))} \quad (25)$$

using the orthogonality relationships in (24). Thus, the differential equation for  $\alpha$  is given by

$$\frac{d\alpha(t)}{dt} = \bar{u}_1^T(t + \alpha(t)) B(x_s(t + \alpha(t))) b(t), \quad \alpha(0) = 0 \quad (26)$$

where

$$\bar{u}_1^T(t) = \frac{u_1^T(t)}{u_1^T(t) u_1(t)} \quad (27)$$

In Section 5.2, we compare the stochastic characterisations of phase noise obtained by the phase error equation (13) based on the correct oblique decomposition, and (26) based on the invalid orthogonal decomposition, and show how orthogonal decomposition can produce completely erroneous results for practical oscillator circuits even though in certain cases it can produce results close to the correct ones. Note that (26) can be obtained from (13) by replacing  $v_1(t)$  with  $\bar{u}_1(t)$ . Hence, the phase error characterisation summarized in Remark 4.1 can also be used for the phase error in (26) by calculating  $c$  in (22) using  $\bar{u}_1(t)$  instead of  $v_1(t)$ . In [DMR97], we present both time-domain and frequency-domain numerical methods for the calculation of the periodic vector  $v_1(t)$  in (22) needed for the characterisation of phase noise. We modified our implementation of the time-domain numerical method to also calculate the periodic vector  $\bar{u}_1(t)$  in (27), so that we can compare the characterisations obtained by the oblique and orthogonal decompositions.

#### 4.1 Linear vs. nonlinear equation for phase error

(26) is similar to an equation derived by the authors in [HL97] for phase error, except for one important difference: (26) is a *nonlinear* equation, whereas the equation derived by the authors in [HL97] is *linear* (i.e.,  $\alpha$  on the RHS of (26) is absent), based on an unjustified superposition argument which assumes that the relationship between the phase error and the perturbation is linear. Kärtner in [Kär90] also derives a *linear* equation for phase error, even though his derivation is based on the correct oblique decomposition. At first sight, the linearisation approximation might seem to be justifiable considering that the perturbations are “small”. However, when one considers random perturbations, i.e., noise which is a small perturbation, one arrives at completely different qualitative results. With the nonlinear phase error equation we obtain the stochastic characterisation summarized in Remark 4.1. On the other hand, if one uses the linear phase error equation, one can show that

- The variance of the phase error has a periodically time-varying component apart from a linearly increasing part, i.e.,

$$\mathbb{E} [\alpha^2(t)] = \sigma^2(t) = ct + p(t)$$

for some periodic  $p(t)$ .

- The time-shifted output of the oscillator, i.e.,  $x_s(t + \alpha(t))$ , with the phase error  $\alpha(t)$  characterised as above, becomes a *cyclostationary* stochastic process asymptotically with time.

**Remark 4.2** *The result in Remark 4.1, based on the nonlinear phase error equation, about oscillator output being stationary might be surprising at first sight, since oscillators are nonlinear systems with periodic swings, hence it might be expected that output noise power would change periodically as in a forced system and as predicted by the characterisation based on the linear phase error equation. However, it must be remembered that while forced systems are supplied with an external time reference (through the forcing), oscillators are not. Cyclostationarity in the oscillator's output would, by definition, imply a time reference. Hence the stationarity result reflects the fundamental fact that noisy autonomous systems cannot provide a perfect time reference. Thus, the result obtained by the linear phase equation is not physical.*

## 5 Examples

### 5.1 $\delta$ -function perturbations

In this section, we present a simple example for the special case of  $\delta$ -function perturbations which will illustrate how

- a perturbation that is orthogonal to the tangent to the limit cycle *can* create phase errors,
- a perturbation that is *not* orthogonal to the tangent to the limit cycle, but that does *not* have a component along the tangent when decomposed using an *oblique* basis (the tangent vector is a component of this basis), does not result in a phase error.

We use the simple oscillator in Figure 2 to illustrate the above.

*Note that an ideal LC oscillator without a resistive loss and without a nonlinearity is not an asymptotically orbitally stable oscillator that satisfies the conditions discussed in Section 2. Many try to build intuition and test their proposed techniques for oscillator analysis using an ideal LC oscillator, which will lead to erroneous results and false intuition about the dynamics of practical oscillators which always have loss and a nonlinearity (active devices form up the nonlinearity that compensates for the loss and maintains a steady oscillation). An LC oscillator is an orbitally unstable system without a stable limit cycle. Even very small perturbations that do not persist can cause an ideal LC oscillator to deviate from its unperturbed limit cycle and never return back to it. On the other hand, all practical oscillators return back to their stable limit cycles after being subject a small perturbation that does not persist.*

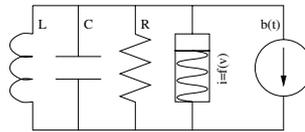


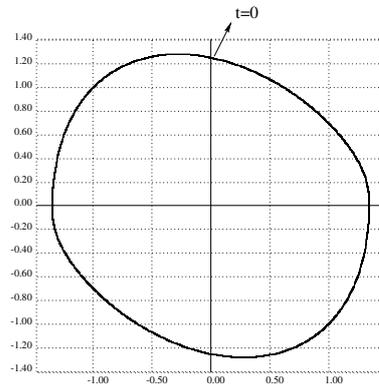
Figure 2: Oscillator with a parallel RLC and a nonlinear current source

This oscillator has two state variables, namely the capacitor voltage  $V$  and the inductor current  $I$ , hence the state-space dimension is 2 with the state vector

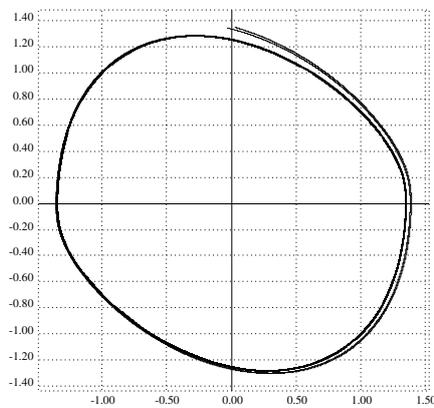
$$x(t) = \begin{bmatrix} V(t) \\ I(t) \end{bmatrix}$$

The limit cycle lies in a plane, called the phase-plane, where one plots  $V(t)$  vs.  $I(t)$  with  $t$  as a parameter. The stable limit cycle for this oscillator is shown in Figure 3(a). We have arbitrarily associated a point on the limit cycle with  $t = 0$ . We apply a  $\delta$ -function perturbation to this oscillator at  $t = 0$ . A  $\delta$ -function perturbation creates an instantaneous change in the state vector, i.e., the capacitor voltage and the inductor current. Instead of applying  $\delta$ -function perturbations, one can equivalently set initial conditions for the capacitor voltage and the inductor current at  $t = 0$ . If the dynamics of the oscillator is described by

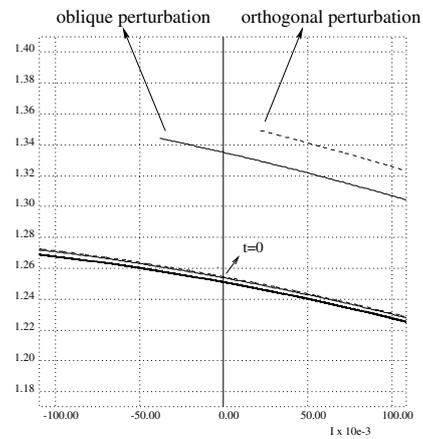
$$\dot{x} = f(x) \tag{28}$$



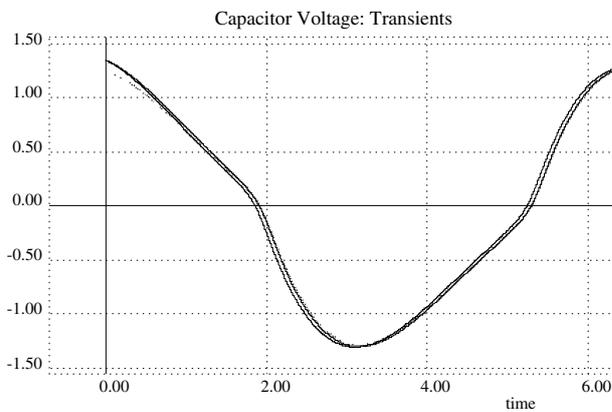
(a) Limit cycle



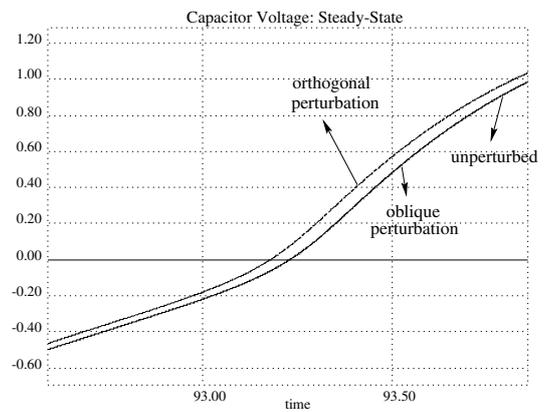
(b) Limit cycle and perturbed trajectories



(c) Orthogonal and oblique perturbations



(d) Cap. voltage: after perturbation, transients



(e) Cap. voltage: after perturbation, steady-state

Figure 3: Orthogonal and oblique  $\delta$ -function perturbations

with the steady-state periodic solution  $x_s(t)$ ,  $-\infty < t < \infty$ , then, for  $t > 0$ , the differential equations

$$\dot{x} = f(x) + \bar{b}\delta(t) \quad x(0) = x_s(0) \quad (29)$$

and

$$\dot{x} = f(x) \quad x(0) = x_s(0) + \bar{b} \quad (30)$$

have the same solution. We now consider two perturbations:

- $b_\perp$  that is orthogonal to  $\dot{x}_s(0)$ , i.e.,

$$b_\perp^T \dot{x}_s(0) = b_\perp^T u_1(0) = 0 \quad (31)$$

- $b_\zeta$  that does not have a component along  $u_1(0) = \dot{x}_s(0)$  when decomposed using the basis vectors  $\{u_1(0), u_2(0)\}$ , i.e.,

$$b_\zeta^T v_1(0) = 0 \quad (32)$$

$\{u_1(0), u_2(0)\}$  and  $\{v_1(0), v_2(0)\}$  are the bases described in Section 2. We set the magnitudes of the perturbations to be equal, i.e.,

$$b_\zeta^T b_\zeta = b_\perp^T b_\perp$$

Figure 3(b) shows three trajectories in the  $V$ - $I$  plane for the solutions ( $t > 0$ ) of (28) for the unperturbed oscillator, and (30) for  $\bar{b} = b_\perp$  and  $\bar{b} = b_\zeta$  for the perturbed oscillator. Figure 3(c) is a blown-up version of Figure 3(b) around  $t = 0$ , showing the orthogonal and oblique perturbations. As expected, the solution traces for the perturbed oscillator eventually return to the limit cycle of the unperturbed oscillator. Figure 3(d) shows the capacitor voltage as a function of time for the unperturbed, and the orthogonally and obliquely perturbed oscillator, right after the perturbations are applied. The capacitor voltages at steady-state are in Figure 3(e). For large  $t$ , the capacitor voltages for the unperturbed and the obliquely perturbed oscillator are indistinguishable, i.e., the perturbation in (32) does not cause any phase error. On the other hand, as seen in Figure 3(e), the orthogonal perturbation in (31) causes a phase error, which can be predicted with

$$\Delta t = b_\perp^T v_1(0) \quad (33)$$

as well as from the numerical solutions of (28) and (30) plotted in Figure 3(e). We obtained the same result using both. Using the phase error theory we developed, we were able to not only show that orthogonal (to the tangent of the limit cycle) perturbations cause phase errors, but also calculate (using (33)) the exact phase error arising from such a perturbation. We illustrated our theory with the simple example above. The SPICE input files for the unperturbed and the perturbed oscillators are provided in Appendix A for the reader who is interested in reproducing the simulations shown for the above example.

## 5.2 Random perturbations: Noise in electrical oscillators

**Oscillator with a parallel RLC and a nonlinear current source** We now present phase noise characterisation results for the simple oscillator in Figure 2. The resistor is assumed to be noiseless, but we insert a stationary white external current noise source across the capacitor.  $v_1(t)$  and  $\bar{u}_1(t)$  of Section 3 are two-dimensional vectors, since the oscillator has two state variables. Figure 4(a) and Figure 4(b) shows the entries of  $v_1^2(t)$  and  $\bar{u}_1^2(t)$  corresponding to the capacitor voltage for two cases:

- (a)  $C = 1, R = 1, L = 1$
- (b)  $C = 0.1, R = 1, L = 1$

In both cases, the nonlinear current source is given by

$$I = \tanh(-5V)$$

Table 1 shows the value of  $c$  in (22) calculated using  $v_1(t)$  and  $\bar{u}_1(t)$  for the two cases above.

In case (a), the orthogonal and oblique decompositions produce ‘‘approximately’’ equal phase noise characterisations due to the fact that the waveforms for  $v_1(t)$  and  $\bar{u}_1(t)$  are ‘‘approximately’’ equal. On the other hand, in case (b), where we change one of the circuit parameters, the oblique and orthogonal decompositions produce different results.

(22) can be rewritten as

$$c = \sum_{i=1}^p \frac{1}{T} \int_0^T [v_1^T(\tau) B_i(\tau)]^2 d\tau = \sum_{i=1}^p c_i \quad (34)$$

	$c$ oblique decomposition	$c$ orthogonal decomposition
(a)	$4.57954 \times 10^{-11}$	$4.28370 \times 10^{-11}$
(b)	$1.48739 \times 10^{-9}$	$5.99340 \times 10^{-9}$

Table 1: Phase noise characterisation with oblique and orthogonal decompositions

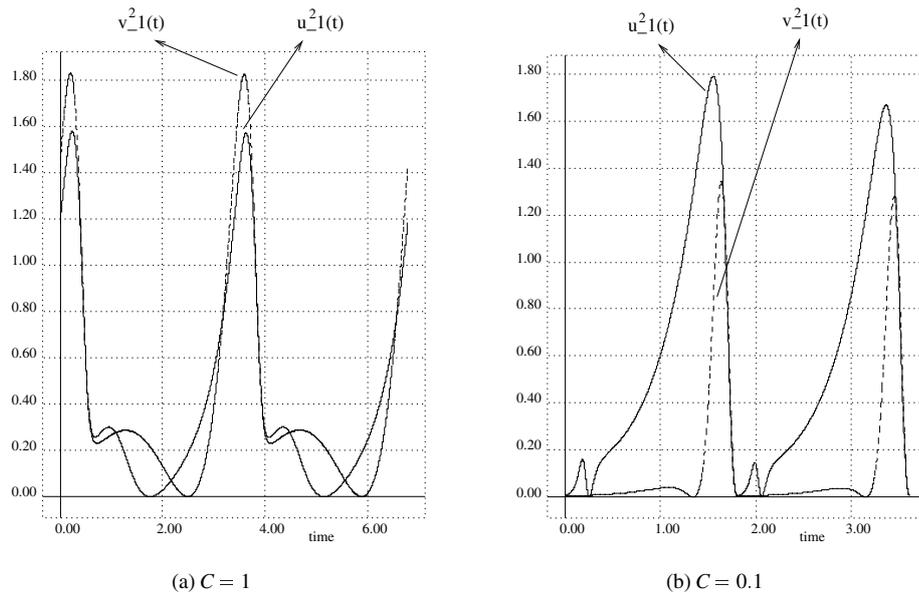


Figure 4:  $v_1^2(t)$  and  $u_1^2(t)$  for the capacitor voltage

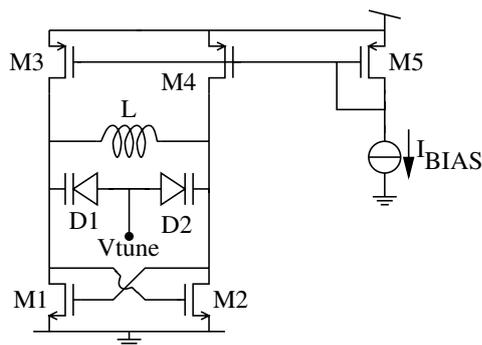


Figure 5: Oscillator with on-chip inductor

where  $p$  is the number of the noise sources, i.e., the column dimension of  $B(x_s(\cdot))$ , and  $B_i(\cdot)$  is the  $i$ th column of  $B(x_s(\cdot))$  which maps the  $i$ th noise source to the equations of the system. Hence,

$$c_i = \frac{1}{T} \int_0^T [v_1^T(\tau) B_i(\tau)]^2 d\tau \quad (35)$$

represents the contribution of the  $i$ th noise source to  $c$ . (35) illustrates how the entries of  $v_1(t)$  ( $\bar{u}_1(t)$  in the case of the orthogonal decomposition) act as sort of a transfer function between the noise source and the scalar  $c$  that characterises phase noise. Notice that the waveforms for  $\bar{u}_1(t)$  and  $v_1(t)$  for the capacitor voltage in case (b) above are quite different. Let us make the noise source connected across the capacitor time-varying, instead of stationary: It turns on briefly with a large amplitude when  $v_1(t)$  is approximately 0 and  $\bar{u}_1(t)$  has a large amplitude. In this case, the oblique decomposition would predict almost no phase noise, whereas the orthogonal decomposition predicts a considerable amount.

With the phase noise characterisations of this simple oscillator we illustrated how:

- for a certain set of parameters for the oscillator circuit (case (a)), the oblique and orthogonal decompositions produce approximately equal phase noise characterisation results,
- changing a parameter in the circuit can result in totally different phase noise characterisations by the oblique and orthogonal decompositions.

**2.5 GHz Oscillator with on-chip inductor** A simplified schematic for this oscillator is in Figure 5. We computed

- $c = 1.34 \times 10^{-19} \text{ sec}^2 \cdot \text{Hz}$  using oblique decomposition
- $c = 4.54 \times 10^{-14} \text{ sec}^2 \cdot \text{Hz}$  using orthogonal decomposition

These values of  $c$  correspond to the following single-sideband phase noise [DMR97] characterisations

- $\mathcal{L}(f_m) = -100.65 \text{ dBc/Hz}$  at  $f_m = 100 \text{ KHz}$  using oblique decomposition
- $\mathcal{L}(f_m) = -45.35 \text{ dBc/Hz}$  at  $f_m = 100 \text{ KHz}$  using orthogonal decomposition
- $\mathcal{L}(f_m) = -96 \text{ dBc/Hz}$  at  $f_m = 100 \text{ KHz}$  measured result for the fabricated circuit

The orthogonal decomposition predicts an unrealistic phase noise characterisation which is 50 dBc/Hz off from the measured one. We believe that the 4.65 dBc/Hz difference between the measured result and the one predicted by the oblique decomposition is due to the following:

- The oscillator circuit simulated was not extracted from the layout.
- The MOS device models and the noise models used in the simulation were not characterised for the process the oscillator was fabricated with.

## A SPICE input files

### Unperturbed oscillator

Parallel RLC with a nonlinearity

```
L x 0 1 IC= 0
C x 0 1 IC= 1.25103
G x 0 x 0 1
B x 0 I=tanh(-5*V(x))

.tran 1 100 0 1e-2 uic
.end
```

### Perturbation orthogonal to the tangent

Parallel RLC with a nonlinearity

```
L x 0 1 IC= 2.156949335175186e-02
C x 0 1 IC= 1.348676080087983e+00
G x 0 x 0 1
B x 0 I=tanh(-5*V(x))
```

```
.tran 1 100 0 1e-2 uic
.end
```

### Perturbation oblique to the tangent

Parallel RLC with a nonlinearity

```
L x 0 1 IC= -3.798952313514651e-02
C x 0 1 IC= 1.343532952017566e+00
G x 0 x 0 1
B x 0 I=tanh(-5*V(x))
```

```
.tran 1 100 0 1e-2 uic
.end
```

## References

- [DMR97] A. Demir, A. Mehrotra, and J. Roychowdhury. Phase noise in oscillators: A unifying theory and numerical methods for characterisation. Technical memorandum, Bell Laboratories, Murray Hill, 1997.
- [HL97] A. Hajimiri and T.H. Lee. A state-space approach to phase noise in oscillators. Technical memorandum, Lucent Technologies, July 1997.
- [Kär90] F. Kärtner. Analysis of white and  $f^{-\alpha}$  noise in oscillators. *International Journal of Circuit Theory and Applications*, 18:485–519, 1990.